B.Sc. (Hons.) Part - II

Paper :- III (Real Analysis)

Dedekind's Theory of real numbers :

Q No - Give Dedekind's theory of real no. Define addition and multiplication of real nos.

Ans: - Dedekind's Section :- Let Q be the set of rational nos. we divide the set of rational nos. into two classes L & R such that :-

(i) $L \subset Q$, $R \subset Q$, $L \neq \phi$, $R = \phi$

(ii) $L \cup R = Q$

(iii) $x \in L$, $y \in R$, then $x < y$ and consequently $L \cap R = \phi$.

Such a pair as (L, R) of subsets of Q which satisfy above three conditions is called section of rational nos. and denoted by (L, R) where L is lower class R is the upper class of the section.

Following are the examples of different types of section,

(i) The lower class L has a greatest member and the upper class R has no smallest member.

$$L = \{x \in Q : x \leq 5\}$$

$$R = \{x \in Q : x > 5\}$$

from above the greatest no. of L is 5 but R has no. smaller no.

(ii) The lower class L has no greatest member and ~~the~~ the upper class R has the smallest member

$L = \{x \in Q : x < 5\}$

$R = \{x \in Q : x \geq 5\}$

(iii) The lower class L has no greatest member and the upper class has no smallest member

$L = \{x \in Q : x < 0 \text{ or } x^2 < 2\}$

$R = \{x \in Q : x^2 > 2, x > 0\}$

We know that $\sqrt{2}$ is not a rational no.

If possible let $\sqrt{2} = \frac{p}{q}$ is a rational no. where p & q is prime to each other,

$\therefore \frac{p^2}{q^2} = 2$

or, $p^2 = 2q^2$ —— (a)

i.e. p^2 is multiple of p

Let $p = 2m$

from (a),

$4m^2 = 2q^2$

or, $q^2 = 2m^2$.

from this it is clear that q^2 is also multiple of 2 cuts. Thus, we see that p & q has a common factor thus, is contradiction. Hence there is no rational no. whose square is 2.

Now, we shall show that L has no greatest member and R has no smallest member.

If possible let K is the greatest member of L.

Then, $K > 0$ and $K^2 < 2$.

Now, Consider the +ve no.

$$\frac{4+3K}{3+2K}$$

We have,

$$2-\left(\frac{4+3K}{3+2K}\right)^2 = \frac{2(3+2K)^2-(4+3K)^2}{(3+2K)^2}$$

$$= \frac{2(9+12K+4K^2)-(16+24K+9K^2)}{(3+2K)^2}$$

$$= \frac{18+24K+8K^2-16+24K+9K^2}{(3+2K)^2}$$

$$= \frac{2-K^2}{(3+2K)^2} > 0, \text{ Since } K^2 < 2.$$

$$\therefore \left(\frac{4+3K}{3+2K}\right)^2 < 2.$$

$\therefore \frac{4+3K}{3+2K} \in L$ and is greater than K.

$\therefore K$ is not the greatest of L and, we have Contradiction.

Similarly it can be shown that R has no smallest no.

<u>Addition of real no:</u> :- In order to define the sum of two cuts (X_1, X_2) & (Y_1, Y_2) we consider the following two cases,

(i) The class Z_2 consisting of all rational no. Z_2

of the form $Z_2 = x_2 + y_2$ where x_2 denote the real no. of ~~X_2 & Y_2, x_2 &~~ X_2 and y_2 any member of Y_2.
(ii) The class Z_1 consisting of all other rational numbers.

$$\therefore (Z_1, Z_2) = (X_1, X_2) + (Y_1, Y_2).$$

Multiplication of Cuts:- Let (X_1, X_2) and (Y_1, Y_2) be any two non-negative cuts. In order to define the Product of (X_1, X_2) and (Y_1, Y_2), we form the class Z_2 consisting of all rational no. of the form $Z_2 = x_2 y_2$, where x_2 & y_2 are members of X_2 & Y_2 respectively and the class Z_1 consisting of all other rational nos. Then it is easily to see that the ordered Pair (Z_1, Z_2) is actually a cut called the Product of the two cuts (X_1, X_2) & (Y_1, Y_2) and we write,

$$(Z_1, Z_2) = (X_1, X_2)\,(Y_1, Y_2).$$